Find the general solution of $y''' + y'' - 2y = 2e^t - 4e^t \cos t$. **SCORE: _____ / 45 PTS**

$r^3 + r^2 - 2 = 0$

$$\begin{array}{r|rrrr} 1 & 1 & 1 & 0 & -2 \\ & & 1 & 2 & 2 \\ \hline & 1 & 2 & 2 & \boxed{0} \end{array}$$

$r^2 + 2r + 2 = 0$

$r = -1 \pm i$

$y_h = c_1 e^t + c_2 e^{-t}\cos t + c_3 e^{-t}\sin t$ (3)

$y_p = Ate^t + Be^t\cos t + Ce^t\sin t$

$y_p' = (At + A)e^t + (B + C)e^t\cos t + (C - B)e^t\sin t$ (3)

$y_p'' = (At + 2A)e^t + 2Ce^t\cos t - 2Be^t\sin t$ (3)

$y_p''' = (At + 3A)e^t + (-2B + 2C)e^t\cos t + (-2C - 2B)e^t\sin t$ (3)

$y_p''' + y_p'' - 2y_p = 5Ae^t + (-4B + 4C)e^t\cos t + (-4C - 4B)e^t\sin t$

$5A = 2$ $\quad -4B + 4C = -4 \quad -B + C = -1$

$A = \frac{2}{5}$ $\quad -4C - 4B = 0 \quad B + C = 0$

$2C = -1$

$C = -\frac{1}{2}$

$B = \frac{1}{2}$

(1) (1) (1)

$y = \frac{2}{5}te^t + \frac{1}{2}e^t\cos t - \frac{1}{2}e^t\sin t$
$+ c_1 e^t + c_2 e^t\cos t + c_3 e^{-t}\sin t$

(2) EACH
EXCEPT AS NOTED

Use elimination (as shown in lecture) to solve the system

SCORE: _____ / 45 PTS

$$(2D+1)[x]-(3D+4)[y]=4t$$
$$(3D+2)[x]-(5D+8)[y]=2t$$

$$x = 4t^2+2t + 2+4c_1 + c_2e^{-3t}$$
$$y = t^2+2t + c_1 + c_2e^{-3t}$$

$$[-(3D+2)(3D+4)+(2D+1)(5D+8)][y] = (3D+2)[4t]-(2D+1)[2t]$$

$$(D^2+3D)[y] = 12+8t-(4+2t) = 8+6t$$

$$r^2+3r=0 \rightarrow r=0,-3$$

$$y_h = c_1+c_2e^{-3t}$$

$$y_p = (At+B)t = At^2+Bt$$

$$y_p' = 2At+B$$

$$y_p'' = 2A$$

$$y_p''+3y_p' = 6At+(2A+3B) = 8+6t$$

$$6A=6 \qquad 2A+3B=8$$
$$A=1 \qquad 2+3B=8$$
$$B=2$$

$$y = t^2+2t+c_1+c_2e^{-3t}$$

$$[(5D+8)(2D+1)-(3D+4)(3D+2)][x] = (5D+8)[4t]-(3D+4)[2t]$$

$$(D^2+3D)[x] = 20+32t-(6+8t) = 14+24t$$

$$x_h = k_1+k_2e^{-3t}$$

$$x_p = Ct^2+Dt$$

$$x_p''+3x_p' = 6Ct+(2C+3D) = 14+24t$$

$$6C=24 \qquad 2C+3D=14$$
$$C=4 \qquad 8+3D=14$$
$$D=2$$

$$x = 4t^2+2t+k_1+k_2e^{-3t}$$

$$\begin{aligned} &2(8t+2-3k_2e^{-3t}) \\ &+(4t^2+2t+k_1+k_2e^{-3t}) \\ &-3(2t+2-3c_2e^{-3t}) \\ &-4(t^2+2t+c_1+c_2e^{-3t}) \end{aligned} = 4t+(k_1-4c_1-2)+(-5k_2+5c_2)e^{-3t}$$

$$k_1-4c_1-2=0 \qquad -5k_2+5c_2=0$$
$$k_1=4c_1+2 \qquad k_2=c_2$$

Find the general solution of $4x^2y'' + 8xy' + y = \dfrac{16}{\sqrt{x}}$.

$g = \dfrac{16x^{-\frac{1}{2}}}{4x^2} = \underline{4x^{-\frac{5}{2}}}$

SCORE: _____ / 30 PTS

$\underline{4r^2 + 4r + 1 = 0}$

$(2r+1)^2 = 0$

NOTE: $x > 0$

$\underline{r = -\frac{1}{2}, -\frac{1}{2}}$

$\underline{y_1 = x^{-\frac{1}{2}},\ y_2 = x^{-\frac{1}{2}}\ln x}$

$W = \begin{vmatrix} x^{-\frac{1}{2}} & x^{-\frac{1}{2}}\ln x \\ -\frac{1}{2}x^{-\frac{3}{2}} & -\frac{1}{2}x^{-\frac{3}{2}}\ln x + x^{-\frac{3}{2}} \end{vmatrix} = \underline{x^{-2}}$

$y_p = \underline{-x^{-\frac{1}{2}}\int \dfrac{4x^{-\frac{5}{2}}x^{-\frac{1}{2}}\ln x}{x^{-2}}\,dx + x^{-\frac{1}{2}}\ln x \int \dfrac{4x^{-\frac{5}{2}}x^{-\frac{1}{2}}}{x^{-2}}\,dx}$

$= -x^{-\frac{1}{2}}\int \underline{4x^{-1}\ln x}\,dx + x^{-\frac{1}{2}}\ln x \int \underline{4x^{-1}}\,dx$

$u = \ln x$

$= -x^{-\frac{1}{2}}(\underline{2(\ln x)^2}) + x^{-\frac{1}{2}}\ln x\,(\underline{4\ln|x|})$ $(x > 0)$

$= 2x^{-\frac{1}{2}}(\ln x)^2$

$y = \underline{2x^{-\frac{1}{2}}(\ln x)^2} + \underline{c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}}\ln x}$

$2\frac{1}{2}$ EACH

$y = x$ is a solution of $x^2y'' + (2x^2 - 2x)y' + (2 - 2x)y = 0$. **SCORE: _____ / 30 PTS**

$y = x^2$ is a particular solution of $x^2y'' + (2x^2 - 2x)y' + (2 - 2x)y = 2x^3$.

Solve the initial value problem $x^2y'' + (2x^2 - 2x)y' + (2 - 2x)y = -8x^3$, $y(1) = 2$, $y'(1) = -14$.

$\uparrow$

$= -4(2x^3)$

SO $y_p = -4x^2$

$y_2 = vx$

$y_2' = v'x + v$

$y_2'' = v''x + 2v'$

$x^2(xv'' + 2v')$
$+(2x^2-2x)(xv' + v)$
$+(2-2x)(xv)$

$= x^3v'' + 2x^3v' = 0$

$v'' + 2v' = 0 \qquad u = v'$

$u' + 2u = 0$

$\frac{1}{u}\,du = -2\,dx$

$\ln|u| = -2x$

$v' = u = e^{-2x}$

$v = -\frac{1}{2}e^{-2x}$

$y_2 = -\frac{1}{2}xe^{-2x}$ OR xe^{-2x}

$y = -4x^2 + Ax + Bxe^{-2x}$

$y' = -8x + A + Be^{-2x} - 2Bxe^{-2x}$

$2 = -4 + A + Be^{-2}$

$-14 = -8 + A + Be^{-2} - 2Be^{-2}$

$= -8 + A - Be^{-2}$

$-12 = -12 + 2A \rightarrow A = 0$ 1

$2 = -4 + Be^{-2} \rightarrow B = 6e^2$

$y = -4x^2 + 6xe^{2-2x}$

1

2 EACH

EXCEPT AS NOTED